

Proof of Schur's conjecture in \mathbb{R}^d

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Abstract

In this paper we prove Schur's conjecture in \mathbb{R}^d , which states that any diameter graph G in \mathbb{R}^d on n vertices may have at most n cliques of size d . We obtain an analogous statement for diameter graphs on a sphere S_r^d of radius $r > 1/\sqrt{2}$. The proof rests on the following statement, conjectured by F. Morić and J. Pach: given two unit regular simplices Δ_1, Δ_2 on d vertices in \mathbb{R}^d , either they share $d - 2$ vertices, or there are vertices $v_1 \in \Delta_1, v_2 \in \Delta_2$ such that $\|v_1 - v_2\| > 1$. The same holds for unit simplices on a d -dimensional sphere of radius greater than $1/\sqrt{2}$.

MSC: 52C10.

Keywords: Schur's conjecture, diameter graph, unit simplices in \mathbb{R}^d .

1 Introduction

One of the classical problems in discrete geometry, raised by P. Erdős in 1946 [6], is the following: given n points in the plane, how many unit distances they may determine? The key definition related to the question of P. Erdős is that of a *unit distance graph*. A graph G is a *unit distance graph* in \mathbb{R}^d if its set of vertices is a finite subset of \mathbb{R}^d and the edges are formed by the pairs of vertices which are at unit distance apart. In terms of distance graphs the question is to determine the maximal number of edges in a planar unit distance graph on n vertices. In this paper we focus on the questions of this type for *diameter graphs*. A graph $G = (V, E)$ is a *diameter graph* in \mathbb{R}^d , if $V \subset \mathbb{R}^d$ is a finite set of diameter 1, and edges of G are formed by the pairs of vertices that are at unit distance apart.

Diameter graphs arise naturally in the context of the finite version of the famous Borsuk's problem (see, e.g., [3, 16] for the survey on Borsuk's problem), which is stated as follows: is it true that any (finite) set of unit diameter in \mathbb{R}^d can be partitioned into $d + 1$ subsets of strictly smaller diameter? The finite

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version is equivalent to the following question concerning diameter graphs: is it true that any diameter graph G in \mathbb{R}^d satisfies $\chi(G) \leq d + 1$?

A question about diameter graphs analogous to the question from the first paragraph has a simple answer: any set of n points in the plane generates at most n diameters, or any diameter graph on n vertices in the plane has at most n edges. This was proved by H. Hopf and E. Pannwitz in [8]. Interestingly, this result leads to a simple proof of the fact that Borsuk's question for finite sets in the plane have a positive answer. Indeed, it is easy to derive combinatorially that any graph G on n vertices with at most n edges and such that any of its subgraphs has at least as many vertices as edges satisfies $\chi(G) \leq 3$. A. Vázsonyi conjectured that any diameter graph in \mathbb{R}^3 on n vertices has at most $2n - 2$ edges. Again, it is easy to see that Borsuk's conjecture for finite sets in \mathbb{R}^3 follows from this statement. Vázsonyi's conjecture was proved independently by B. Grünbaum [9], A. Heppes [10] and S. Straszewicz [18]. An interesting generalization of this result to the case of k -th diameters was obtained by F. Morić and J. Pach [15].

While the maximum number of edges in a diameter graph in $\mathbb{R}^2, \mathbb{R}^3$ is linear in the number of vertices, it becomes quadratic already in \mathbb{R}^4 . To put the discussion in a more general context, we introduce the following notations. Denote by $D_d(l, n)$ ($U_d(l, n)$) the maximum number of cliques of size l in a diameter (unit distance) graph on n vertices in \mathbb{R}^d . P. Erdős [6, 7] studied $U_d(2, n)$ and $D_d(2, n)$ for different d . He showed that for $d \geq 4$ we have $U_d(2, n), D_d(2, n) = \frac{\lfloor d/2 \rfloor - 1}{2 \lfloor d/2 \rfloor} n^2 + o(n^2)$. Swanepoel [19] determined $U_d(2, n)$ for fixed even $d \geq 6$ and sufficiently large n depending on d and determined $D_d(2, n)$ for $d \geq 4$ and sufficiently large n .

Functions $D_d(l, n)$, $U_d(l, n)$ for $l > 2$ and similar functions were studied in several papers. In particular, the following conjecture was raised in [17]:

Conjecture 1 (Schur et. al., [17]). *We have $D_d(d, n) = n$ for $n \geq d + 1$.*

This was proved by H. Hopf and E. Pannwitz for $d = 2$ in [8] and for $d = 3$ by Z. Schur et. al. in [17]. In the latter paper the authors also proved that $D_d(d + 1, n) = 1$. In [14] P. Morić and J. Pach progressed towards resolving this conjecture. Namely, they showed that Schur's conjecture holds in the following special case:

Theorem 2 (Theorem 1 from [14]). *Given a diameter graph G on n vertices in \mathbb{R}^d , the number of d -cliques in G does not exceed n , provided that any two d -cliques share at least $d - 2$ vertices.*

As it turns out, Schur's conjecture and related questions are tightly connected with analogous questions for spherical sets. The spherical analogues were studied in some papers. In particular, in the paper [4] V. Bulankina et al. noted that the statement of Theorem 2 holds for spheres of large radii: given a diameter graph G on n vertices in a d -dimensional sphere S_r^d with radius $r > 1/\sqrt{2}$, the number of d -cliques in G does not exceed n , provided that any two d -cliques share at least $d - 2$ vertices (Theorem 4 from [4]). Moreover, they

showed that Schur's conjecture holds for S_r^3 for $r > 1/\sqrt{2}$. To be precise, we formulate Schur's conjecture for spheres separately:

Conjecture 3 (Schur's conjecture for spheres). *Any diameter graph G on n vertices (and with edges of unit Euclidean length) on a sphere S_r^d with $r > 1/\sqrt{2}$ has at most n d -cliques.*

In the paper [11] A. Kupavskii studied properties of diameter graphs in \mathbb{R}^4 . The following theorem completes the description of the quantity $D_4(l, n)$ for different l :

Theorem 4 (Theorem 5 from [11]).

1. For $n \geq 52$ we have

$$D_4(2, n) = \begin{cases} \lceil n/2 \rceil \lfloor n/2 \rfloor + \lceil n/2 \rceil + 1, & \text{if } n \not\equiv 3 \pmod{4}, \\ \lceil n/2 \rceil \lfloor n/2 \rfloor + \lceil n/2 \rceil, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

(In Corollary 3 from [19] the same was proved for sufficiently large n .)

2. For all sufficiently large n we have

$$D_4(3, n) = \begin{cases} (n-1)^2/4 + n, & \text{if } n \equiv 1 \pmod{4}, \\ (n-1)^2/4 + n - 1, & \text{if } n \equiv 3 \pmod{4}, \\ n(n-2)/4 + n, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

3. (Schur's conjecture in \mathbb{R}^4) For all $n \geq 5$ we have $D_4(4, n) = n$.

In [11] the first author also studied diameter graphs on S_r^3 with $r > 1/\sqrt{2}$. In particular, he showed that an analogue of Vázsonyi's conjecture holds for spheres.

In the next section we present our main results and discuss related questions. In Section 3 we introduce the basic objects that are used in the proof. In Section 4 we present the proofs of the results.

2 New results and discussion

The main result of this paper is the proof of Schur's conjecture both in the Euclidean space and on the sphere in the general case:

Theorem 5. *Schur's conjecture holds*

1. In the space \mathbb{R}^d ,
2. On the sphere S_r^d of radius $r > 1/\sqrt{2}$.

The proof of the first part actually relies heavily on the second part, so the questions for the Euclidean space and for the sphere are indeed interconnected.

Remark Note that throughout the article by a k -simplex in \mathbb{R}^d we mean a set of $k + 1$ vertices in \mathbb{R}^d in general position.

Next we discuss several questions mentioned in the paper [14]. Since the authors of [14] proved Theorem 2, they naturally raised the following problem:

Conjecture 6 (F. Morić and J. Pach, Problem 1 from [14]). *Any two unit regular simplices on d vertices in \mathbb{R}^d must share at least $d - 2$ vertices, provided the diameter of their union is 1.*

We follow the approach suggested by F. Morić and J. Pach and confirm this conjecture (and its spherical version) in our paper, which together with Theorem 2 and its spherical analogue from [4], mentioned in the previous section, gives us the proof of Schur's conjecture both in the space and on the sphere. Another problem the authors of [14] raised deals with irregular simplices.

Conjecture 7 (Conjecture 3 from [14]). *Let a_1, \dots, a_d and b_1, \dots, b_d be two simplices on d vertices in \mathbb{R}^d with $d \geq 3$, such that all their edges have length at least 1. Then there exist $i, j \in \{1, \dots, d\}$ such that $\|a_i - b_j\| \geq 1$.*

By slightly modifying the proof of Theorem 5 it is not difficult to obtain the following theorem:

Theorem 8. *Consider a regular unit simplex $\{a_1, \dots, a_d\}$ and a simplex $\{b_1, \dots, b_d\}$ in \mathbb{R}^d (or on S_r^d with $r > 1/\sqrt{2}$), where the second simplex satisfies the property $\|b_i - b_j\| \geq 1$ for $i \neq j$. Then either these two simplices share $d - 2$ vertices, or $\|a_i - b_j\| > 1$ for some $i, j \in \{1, \dots, d\}$.*

This theorem solves Conjecture 7 in a stronger form in the case where one of the two simplices is regular. We omit the proof, but the main additional ingredient needed is that the radius of the smallest ball that contains the simplex b_1, \dots, b_d is at least as big as for a regular unit $(d - 1)$ -simplex, provided that $\|b_i - b_j\| \geq 1$ for $i \neq j$. This, in turn, is an easy application of Kirschbraun's theorem (see [1] for a short and nice proof):

Theorem 9 (Kirschbraun's theorem). *Let U be a subset of X , where X is \mathbb{R}^d , S^d or H^d (a d -dimensional hyperbolic space). Then any nonexpansive map $f : U \rightarrow X$ can be extended to a nonexpansive map $f' : X \rightarrow X$. A nonexpansive map $f : Y \rightarrow X$ is a map which satisfies $\|f(a) - f(b)\| \leq \|a - b\|$ for any $a, b \in Y$.*

After having prepared the first version of this paper, we came across a paper by H. Maehara [13], in which the author studies a seemingly unrelated concept of sphericity of a graph: given a graph G , the sphericity of G is the minimum dimension in which the vertices of the graph can be represented as unit spheres in such a way that two spheres intersect (or touch) iff the corresponding vertices are connected by an edge. In the paper [13] the author discusses the sphericity

of complete bipartite graphs. And, as it turned out, the main result of the paper is, in fact, the proof of Conjecture 7, 20 years before the conjecture was formulated! For a bit more on Maehara's result in the context of Schur's conjecture see Paragraph 4.1.

Finally, in the paper [14] the authors raised the following general problem:

Problem 10 (Problem 6 from [14]). *For a given d , characterize all pairs k, l of integers such that for any set of k red and l blue points in \mathbb{R}^d we can choose a red point r and a blue point b such that $\|r - b\|$ is at least as large as the smallest distance between two points of the same color.*

For $k = d + 1$ and $l = \lfloor \frac{d+1}{2} \rfloor$ it is not difficult to construct an example of two regular unit simplices in \mathbb{R}^d on k and l vertices respectively, such that the distance between any two vertices from different simplices is smaller than 1, which we describe at the end of the next section. (An analogous, but somewhat different, example appeared in the latter version of the paper [14].) We think that in this case one cannot take a larger l , thus, we conjecture the following.

Conjecture 11. *Given two unit simplices in \mathbb{R}^d , one on $d + 1$ vertices, the other on $\lfloor \frac{d+1}{2} \rfloor + 1$ vertices, either they share a vertex, or the diameter of their union is strictly larger than 1.*

3 Preliminaries

Given a hyperplane π , we denote by π^+ and π^- two closed half-spaces (half-spheres in the spherical case) that are determined by π .

The following object is very important for understanding diameter graphs:

Definition 1. *A Reuleaux simplex Δ in \mathbb{R}^d is a set formed by the intersection of the balls $B_i = B_1^d(v_i)$ of unit radius with centers in v_i , $i = 1, \dots, d + 1$, where v_i 's are the vertices of a unit simplex in \mathbb{R}^d . In the case $d = 3$ we call this object a Reuleaux tetrahedron, and in the case $d = 2$ we call it a Reuleaux triangle.*

We denote the $(d - 1)$ -dimensional spheres of unit radii with centers in v_1, \dots, v_{d+1} (the boundary spheres of B_1, \dots, B_{d+1}) by S_1, \dots, S_{d+1} . A Reuleaux simplex is a spherical polytope, so one can naturally partition the boundary of a Reuleaux simplex into spherical faces of different dimensions: the vertices of the underlying simplex are the zero-dimensional faces, the arcs that connect the vertices are the one-dimensional faces and so on. We discuss it in more details a bit later in this section. The analogous definition could be given in the case of S_r^d , $r > 1/\sqrt{2}$. In this case we call the body a *spherical Reuleaux simplex*. The only thing one have to keep in mind is that on a d -dimensional sphere we still consider spherical Reuleaux simplices on $d + 1$ vertices. Note that, by Jung's theorem, on a d -dimensional sphere of radius $r = \sqrt{(d + 1)/(2d + 4)}$ one can have a regular unit $(d + 2)$ -simplex, which is, however, impossible for other radii (and, in particular, impossible for $r > 1/\sqrt{2}$).

For a given set W we denote its interior by $\text{int } W$. In the paper we use several times the following simple observation.

Observation 12. Consider two d -balls B, B' in \mathbb{R}^d of radii r, r' , correspondingly. Denote their boundary spheres by S, S' . Assume that $r > r'$ and that S and S' intersect in a $(d - 2)$ -dimensional sphere. Denote the hyperplane that contains $S \cap S'$ by σ and assume that the centers of S, S' are in the same closed halfspace σ^+ with respect to σ . The other halfspace we denote by σ^- . Then $B \cap \sigma^+ \supset B' \cap \sigma^+$ and $B \cap \sigma^- \subset B' \cap \sigma^-$. Moreover, $\text{int}(B \cap \sigma^+) \supset B' \cap \text{int}(\sigma^+)$ and $B \cap \text{int}(\sigma^-) \subset \text{int}(B' \cap \sigma^-)$.

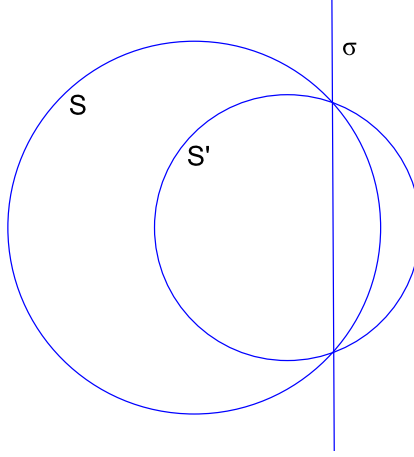


Figure 1

See Fig. 1, illustrating the observation. We apply the observation above to deduce the following two lemmas.

Lemma 13. Consider a Reuleaux simplex $\Delta \subset \mathbb{R}^d$ with the set of vertices v_i , $i = 1, \dots, d + 1$ and a ball B with a boundary sphere S , circumscribed around the d -simplex $\{v_1, \dots, v_{d+1}\}$. Then Δ lies inside B , moreover, $\Delta \cap S = \{v_1, \dots, v_{d+1}\}$.

Proof. Denote by π_i the hyperplane that passes through the vertices $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{d+1}$. We denote the closed halfspace defined by π_i and that contains v_i by π_i^+ . This halfspace contains the convex hull of the vertices $\{v_1, \dots, v_{d+1}\}$. By π_i^- we denote the other halfspace. Applying the part of Observation 12 about interiors to the balls B_i, B , we get that $\text{int } B \supset \text{int}(B \cap \pi_i^-) \supset B_i \cap \text{int}(\pi_i^-)$. After going through all possible values of i , we get that

$$\text{int } B \supset (\cap_{i=1}^{d+1} B_i) \setminus (\cap_{i=1}^{d+1} \pi_i^+) = \Delta \setminus (\cap_{i=1}^{d+1} \pi_i^+).$$

On the other hand, $\cap_{i=1}^{d+1} \pi_i^+$ is just a convex hull of the points $\{v_1, \dots, v_{d+1}\}$, and it is for sure contained in B , moreover, it intersects S only in its vertices v_1, \dots, v_{d+1} . \square

Lemma 14. Consider a Reuleaux simplex $\Delta \subset \mathbb{R}^d$ with the set of vertices v_i , $i = 1, \dots, d + 1$. Then the intersection of Δ with the hyperplane π that passes through v_1, \dots, v_d is a Reuleaux simplex Δ' with vertices $v_i, i = 1, \dots, d$.

Proof. We have $\Delta' = \pi \cap \bigcap_{i=1}^{d+1} B_i$ and we have to prove that $\Delta' = \pi \cap \bigcap_{i=1}^d B_i$. Thus, it is enough to show that $\pi \cap \bigcap_{i=1}^d B_i \subset B_{d+1} \cap \pi$. Denote the circumscribed ball of the Reuleaux simplex $\pi \cap \bigcap_{i=1}^d B_i$ by B' and its boundary sphere by S' . Then $B_{d+1} \cap \pi = B'$, and the statement of the lemma follows from the first claim of Lemma 13. \square

We as well need some knowledge about the structure of a Reuleaux simplex Δ as a spherical polytope. Let the vertices of Δ be v_1, \dots, v_{d+1} and their convex hull be denoted by T . The boundary of the Reuleaux simplex Δ can be partitioned into relatively open spherical regions of different dimensions in the following way. Consider all spheres U that are formed as intersections of several spheres out of S_1, \dots, S_{d+1} . For example, the intersection of the first d spheres is a two-point set, with one of its points being v_{d+1} . We only exclude the intersection of all the spheres, which is empty. We denote the set of all such spheres U by \mathcal{S} . For each point on the boundary of Δ we may find the sphere from \mathcal{S} of minimal dimension that contains it. The set of all points from the boundary of Δ that correspond to a given $U \in \mathcal{S}$ we call a face. It is defined by the set of strict quadratic inequalities and, thus, is naturally a relatively open set.

The center of each sphere $U \in \mathcal{S}$ coincides with the center of some of the faces of T . Namely, if $U = \cap_{j=1}^l S_{i_j}$, where $1 \leq i_1 < i_2 < \dots < i_l \leq d+1$, then the center of U is the centerpoint of the $(l-1)$ -dimensional face with the vertices v_{i_1}, \dots, v_{i_l} . On the other hand, U contains all the vertices from $\{v_1, \dots, v_{d+1}\} \setminus \{v_{i_1}, \dots, v_{i_l}\}$ and, since U is a $(d-l)$ -dimensional sphere and points $\{v_1, \dots, v_{d+1}\}$ are in general position, these points determine U . We call these points the vertices of the face.

Each face of Δ in the sense above is a connected set, moreover, it is obtained from the face of T with the same set of vertices via projection from the center O of T on the boundary of Δ . We verify this property in what follows. Fix a vertex set of the face F of Δ . W.l.o.g., it is $\{v_1, \dots, v_i\}$. The points of F on the boundary of Δ are the ones that, first, are at distance 1 from v_{i+1}, \dots, v_{d+1} and, second, are at distance strictly less than 1 from v_1, \dots, v_i . Similarly to what we have said in the previous paragraph, it follows from the first condition that all these points lie in the flat α , which is an affine hull of the points O, v_1, \dots, v_i (we may use the centerpoint of the face $\{v_{i+1}, \dots, v_{d+1}\}$ of T instead of O). Since Ov_j has the same length for all j , the second condition is equivalent to the fact that each point w from F satisfies the angular inequality $\angle wOv_j \leq \angle wOv_{i+1}$ for all $j = 1, \dots, i$. Each of these inequalities is satisfied in a halfspace, defined by the hyperplane that passes through O and vertices $\{v_1, \dots, v_{d+1}\} \setminus \{v_j, v_{i+1}\}$. These halfspaces bound the face $\{v_1, \dots, v_i\}$ in $T \cap \alpha$ and, as we have showed above, they also bound the face with the same set of vertices in $\Delta \cap \alpha$. Moreover, they pass through O , which concludes the proof of the statement about the central projection. We formulate the findings of the last three paragraphs in a lemma:

Lemma 15. *In the notations introduced above, consider a Reuleaux simplex*

$\Delta \subset \mathbb{R}^d$. Then its boundary may be split into relatively open connected spherical regions (faces), each of which corresponds to an intersection of several spheres out of S_1, \dots, S_{d+1} . The face F of Δ that corresponds to the intersection of the spheres S_{i+1}, \dots, S_{d+1} lies on the sphere with the center in the centerpoint C of the (relatively open) face F' of T with vertices v_{i+1}, \dots, v_{d+1} : $C = (v_{i+1} + \dots + v_{d+1}) / (d - i + 1)$. The minimal flat that contains F , is an affine hull of points O, v_1, \dots, v_i . Moreover, the face F is equal to the central projection of F' from the center of Δ to the boundary of Δ .

Next we define the object which is of a particular importance for the paper:

Definition 2. A rugby ball Θ in \mathbb{R}^d is a set formed by the intersection of the balls $B_i = B_1^d(v_i)$ of unit radius with centers in v_i , $i = 1, \dots, d$, where v_i 's are the vertices of a unit $(d - 1)$ -simplex in \mathbb{R}^d .

We omit the analogous definition of a *spherical rugby ball*. Note the difference between a Reuleaux simplex and a rugby ball. In the latter case, we consider the intersection of d balls instead of $d + 1$ in the former. The intersection of the hyperplane π that passes through v_1, \dots, v_d , and the corresponding rugby ball is a Reuleaux simplex of codimension 1. The rugby ball is symmetric with respect to π .

Moreover, consider a Reuleaux simplex Δ on the vertices v_1, \dots, v_{d+1} , the rugby ball Θ on the vertices v_1, \dots, v_d , and the hyperplane π containing vertices v_1, \dots, v_d . Suppose that $v_{d+1} \in \pi^+$. In the following lemma we prove that $\Delta \cap \pi^+ = \Theta \cap \pi^+$. We denote this body by Δ^+ .

Lemma 16. In the notations introduced above, we have $\Delta \cap \pi^+ = \Theta \cap \pi^+$.

Proof. Since $\Delta \cap \pi^+ = \Theta \cap \pi^+ \cap B_{d+1}$, it is obviously sufficient to show that $B_{d+1} \cap \pi^+ \supset \Theta \cap \pi^+$. Consider a ball B circumscribed around Δ . By using the same argument as in Lemma 13, we get that $B \cap \pi^+ \supset \Theta \cap \pi^+$. On the other hand, applying Observation 12 to B, B_{d+1} , we get that $B_{d+1} \cap \pi^+ \supset B \cap \pi^+$. \square

Now we describe the construction mentioned in the end of the previous section. Take a regular simplex on $d + 1$ vertices in \mathbb{R}^d as the red points. Next, construct the Reuleaux simplex on the the red points and choose $l = \lfloor \frac{d+1}{2} \rfloor$ midpoints y_1, \dots, y_l of some l pairwise disjoint arcs that connect the vertices of the Reuleaux simplex. It could be checked that the distance between the midpoints of two such arcs is strictly bigger than 1. To see this, one have to consider a coordinate representation of the simplex $\{v_1, \dots, v_{d+1}\}$ in the hyperplane $x_1 + \dots + x_{d+1} = 1$ in \mathbb{R}^{d+1} and calculate the coordinates of a middle of an arc that connects two vertices of the corresponding Reuleaux simplex. Thus, if we consider the simplex on y_1, \dots, y_l and contract it a little, we will get a simplex on vertices x_1, \dots, x_l with all vertices inside the Reuleaux simplex and with all sides greater than 1. We take x_i as the blue points, which together with the red points gives us the desired example.

4 Proofs

4.1 Reduction to the auxiliary theorem

The proof of Theorem 5 is based on induction and the following auxiliary theorem, which is of interest by itself:

Theorem 17. *Given a diameter graph G*

1. *In the space \mathbb{R}^d , $d \geq 3$;*
2. *On the sphere S_r^d of radius $r > 1/\sqrt{2}$, $d \geq 3$, any two d -cliques in G must share a vertex.*

As we came across the paper [13] we were thinking whether or not to try to give a proof of Theorem 17 using Maehara's result. The Euclidean case of Theorem 17 is almost equivalent to Theorem 2 from [13] (which is equivalent to Conjecture 7). However, to get part 1 of Theorem 17 from Conjecture 7, one has to replace the non-strict inequality on distances by a strict one. It turns out that this seemingly technical detail is not easy to overcome. If applied directly, Conjecture 7 says only that, if there are two regular unit $(d-1)$ -simplices in a diameter graph in \mathbb{R}^d that do not share a vertex, then there must be at least one edge between them. This is clearly not sufficient (and it is fairly easy to reduce Theorem 17 to this case). The proof of Maehara, however, may be modified to give a proof of Theorem 17 in the Euclidean case, but becomes significantly more complicated. As for the spherical case, we do not even know. It may also be the case that it is possible to apply Maehara's technique, but we believe that in sum it would not simplify the proof of Theorem 17. Therefore, we decided to leave the proof as it is and not to utilize Maehara's ideas.

In this subsection we describe how to derive Theorem 5 from Theorem 17. Consider two d -cliques in a diameter graph G in \mathbb{R}^d (or on S_r^d with $r > 1/\sqrt{2}$). Then, by Theorem 17, these two cliques must share a vertex. All the remaining vertices of the two simplices must lie on the $(d-1)$ -dimensional unit sphere S with the center in the common vertex of the two simplices. The vertices on S form two $(d-2)$ -dimensional unit simplices, and, since the graph on the sphere is still a diameter graph, we can again apply Theorem 17 and obtain that they must share another vertex.

Finally, when we obtain that any two d -cliques must share $d-2$ common vertices we apply a spherical analogue of Theorem 2 from [4], and Schur's conjecture is proved. We only have to verify the following: the spheres that we obtain in such a process always have radius greater than $1/\sqrt{2}$. This was shown to be true in the paper [4] (Lemma 4). We state this fairly easy lemma and present its proof for completeness.

Lemma 18. *Consider a d -dimensional sphere $S = S_r^d$ of radius $r > 1/\sqrt{2}$ and a unit simplex Δ on k vertices v_1, \dots, v_k with all its vertices on S . Then the intersection Ω of the sphere S and the k unit spheres with centers in v_1, \dots, v_k is a sphere of radius $r_\Omega > 1/\sqrt{2}$.*

Proof. We assume that the sphere is embedded into the Euclidean space, and we work in that space. Denote by $v = \frac{1}{k} \sum_{i=1}^k v_i$ the center of the sphere S' , circumscribed around Δ . By Jung's theorem, the radius r' of S' is equal to $\sqrt{\frac{k-1}{2k}}$. So, the radius r'' of the sphere S'' , which is the intersection of k unit spheres with centers in v_1, \dots, v_k is $\sqrt{1 - \frac{k-1}{2k}} = \sqrt{\frac{k+1}{2k}}$. Note that the center of S'' is also v . Denote the center of S by O . Then the center w of Ω lies on the segment Ov of length b . Since v_1, \dots, v_k lie on S , we have $b^2 = r^2 - (r')^2 = r^2 - \frac{k-1}{2k}$. Suppose w splits the segment Ov into the parts of length $b - a, a$ respectively. Then, since $\Omega \subset S$, we get $r_\Omega^2 = r^2 - (b - a)^2$. We also have $\Omega \subset S''$ so we get $r_\Omega^2 = \frac{k+1}{2k} - a^2$. Therefore,

$$2r_\Omega^2 = r^2 - b^2 + \frac{k+1}{2k} + 2ab - 2a^2 = 1 + 2a(b - a) > 1,$$

because it is easy to see that $a, b - a > 0$. \square

Our main goal is to prove Theorem 17. The proof of this theorem also goes by induction. The statement of the theorem is known to be true for \mathbb{R}^3 and S_r^3 for $r > 1/\sqrt{2}$ (see the papers [5], [11]). In what follows we reduce the problem for the d -dimensional space or for the d -dimensional sphere to the analogous problem for the $(d - 1)$ -dimensional sphere. Since the base of the induction is already verified, this concludes the proof of the theorem.

It is tempting to give a unified proof of Theorem 17, in which the Euclidean and the spherical cases are both treated at the same time. But, on the other hand, it would make the proof more difficult to understand, so we decided to give a proof for the Euclidean case first, and then to describe the differences and peculiarities of the spherical case in a separate subsection.

4.2 Proof of Theorem 17. Euclidean case

We begin with the following important lemma:

Lemma 19. *Take a Reuleaux simplex Δ in \mathbb{R}^d and the hyperplane π containing the vertices v_1, \dots, v_d of Δ . Consider the body Δ^+ . Suppose $v, w \in \Delta^+$, and suppose that the projection v' of v on the hyperplane π lies inside the convex hull T of v_1, \dots, v_d . Then $\|v - w\| \leq 1$, with the equality possible in the following two cases: 1. One of the vertices v, w coincides with one of v_1, \dots, v_d . 2. The vertex w lies in the hyperplane π on the border of a Reuleaux simplex Δ_π , constructed on the vertices v_1, \dots, v_d . At the same time the projection v' of the vertex v on the hyperplane π must lie on ∂T .*

Proof. It is enough to consider the case when none of v, w coincide with the vertices of Δ . Consider the projections v', w' of v, w on the hyperplane π (see Fig. 2). We have two possibilities:

1. $\|w' - w\| \geq \|v' - v\|$. Since v' lies inside T and the maximum of the distances from a fixed point to the points of a polytope is attained on the vertices

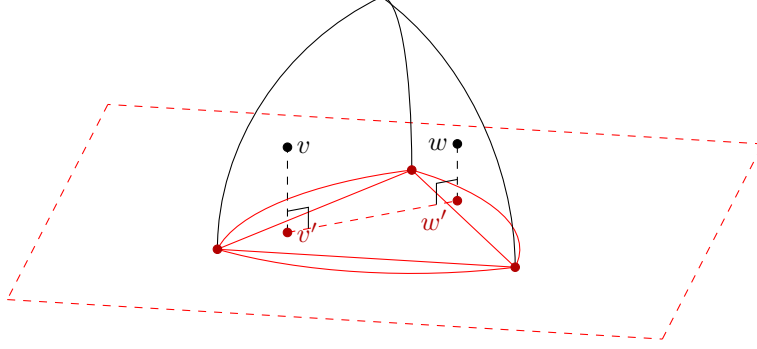


Figure 2

of the polytope, there exists a vertex of T , say, v_1 , such that $\|v_1 - w'\| > \|v' - w'\|$, and we obtain the following chain of inequalities:

$$\begin{aligned} \|v - w\|^2 &= \|w' - v'\|^2 + (\|w' - w\| - \|v' - v\|)^2 \leq \\ &\leq \|w' - v'\|^2 + \|w' - w\|^2 < \|w' - v_1\|^2 + \|w' - w\|^2 = \|w - v_1\|^2 \leq 1. \end{aligned}$$

The proof of the lemma in this case is complete.

2. $\|w' - w\| < \|v' - v\|$. Assume for a moment that we obtained the inequality (in this case not strict) similar to the one in the previous case: there exists a vertex of T , say, v_1 , such that $\|v_1 - v'\| \geq \|v' - w'\|$. Then we obtain the statement of the lemma from the similar chain of inequalities:

$$\begin{aligned} \|v - w\|^2 &= \|w' - v'\|^2 + (\|w' - w\| - \|v' - v\|)^2 \stackrel{(1)}{\leq} \\ &\leq \|w' - v'\|^2 + \|v' - v\|^2 \stackrel{(2)}{\leq} \|v' - v_1\|^2 + \|v' - v\|^2 = \|v - v_1\|^2 \leq 1. \end{aligned}$$

Unfortunately, we do not have any strict inequality in this chain. However, it is clear that the inequality (1) is strict unless $w = w'$. Moreover, it is easy to show that the inequality (2) is strict if w is not on the border of Δ_π , which proves that the equality may hold only if the first part of condition 2 from the lemma is satisfied. Indeed, if w is in Δ_π , but not on the border, then change w to the point of intersection of the ray $v'w$ with the border of Δ_π . The distance between v and w will increase, which makes the inequality strict.

Therefore, to conclude the proof of Lemma 19 it is sufficient to show that such a vertex v_1 exists (and to verify that the part of condition 2 from the lemma concerning v' holds as well). We formulate it as a separate lemma:

Lemma 20. *Consider a Reuleaux simplex Δ_π in \mathbb{R}^{d-1} with the set of vertices v_1, \dots, v_d and two points $v', w \in \Delta_\pi$, different from the vertices of Δ_π , so that v' lies in the convex hull T of v_1, \dots, v_d . Then there exists i so that $\|v_i - v'\| \geq \|w - v'\|$, with the equality possible only in case if v' lies on the boundary of T .*

Lemma 20 is proved using a repetitive application of yet another lemma:

Lemma 21. Consider a closed half-space ω^+ in \mathbb{R}^d bounded by a hyperplane ω . Let Υ be a sphere with center in C , where $C \in \omega$. Let Ω be an open region on Υ , $\Omega \subset \Upsilon \cap \omega^+$. Consider two points $X \in \omega^+$, $Y \in \Omega$. Then one can find a point $Y' \in \partial\Omega$ such that

- $\|X - Y\| < \|X - Y'\|$, if $X \neq C$;
- $\|X - Y\| = \|X - Y'\|$, if $X = C$.

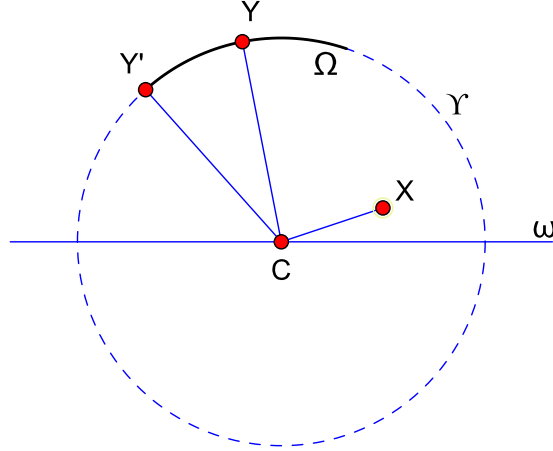


Figure 3

Proof. The equality from the statement of the lemma is obvious since Ω and $\partial\Omega$ both lie on Υ . As for the inequality, consider the two-dimensional plane γ that contains the points C, X, Y (See Fig. 3). The line CY splits the plane into two closed halfplanes γ^+, γ^- . Let $X \in \gamma^+$. There is at least one point $Y' \in \partial\Omega$ in γ^- , which is different from Y . Then we have the inequality for the angles $\angle XCY < \angle XCY'$ and, thus, $\|X - Y\| < \|X - Y'\|$. \square

As we have already said, we apply Lemma 21 repeatedly. We may assume that $w \in \partial\Delta_\pi$. The boundary of a Reuleaux simplex can be partitioned into the open spherical regions of different dimensions, as it is stated in Lemma 15.

We first find an open spherical region Ω that contains w and the sphere U of minimal dimension that contains Ω . We denote its center by C . Then we project T on the flat σ of minimum dimension that contains U . Assume that σ is an affine hull of the points O, v_{k+1}, \dots, v_d (see Lemma 15). Then $C = (v_1 + \dots + v_k)/k$. The projection is fairly simply arranged. Any of the points v_1, \dots, v_k project into C . This is true due to the fact that for any $i = 1, \dots, k$ we have $S_i \cap \sigma = U$. To the contrary, all the points v_{k+1}, \dots, v_d project to themselves, since they lie in σ . It is clear that the projection v'' of v' falls into the projection of T .

In σ consider the hyperplane ω which passes through C and is parallel to the hyperplane that contains v_{k+1}, \dots, v_d . The vertices w, v'' lie in the halfspace

ω^+ in which the whole projection of T lie (recall that $w \in \Delta_\pi$). Now it is possible to apply Lemma 21 and find a point w'' on an open spherical region of a smaller dimension such that $\|v'' - w''\| \geq \|v'' - w\|$. Moreover, if $v' \notin \partial T$, then v'' cannot coincide with C , which means that the inequality is strict. By Pythagoras' theorem we get $\|v' - w''\| \geq \|v' - w\|$ (and a strict inequality in the case when $v' \notin \partial T$). Reducing the dimension of the spherical region which contains the current image of w step by step, we eventually arrive at a vertex of T , which concludes the proof of both Lemma 20 and 19. \square

Consider a diameter graph G and two d -cliques K_1, K_2 in G . Denote by v_1, \dots, v_d the vertices of K_1 . Form a rugby ball Θ on K_1 and denote the hyperplane containing K_1 by π . The following step is essential for the proof. Consider a d -dimensional ball B , circumscribed around the clique K_1 (it has a center in the center O of the clique K_1 and radius $\sqrt{(d-1)/2d}$ but has one dimension more than a normal circumscribed ball). Denote the boundary sphere of B by S . As usually, denote the $(d-1)$ -dimensional spheres of unit radii with centers in v_1, \dots, v_d (the boundary spheres of B_1, \dots, B_d) by S_1, \dots, S_d .

The set $S \cap S_i$ for any $i = 1, \dots, d$ is a sphere that lies in the hyperplane π_i orthogonal to π . Indeed, it is true due to the fact that both centers (O and v_i) lie in π . This together with Observation 12 gives us the following crucial observation.

Observation 22. *In the notations introduced above, whenever a point lies in $\Theta \setminus \text{int } B$, its projection on the hyperplane π falls inside the convex hull T of v_1, \dots, v_d . If a point lies in $\Theta \setminus B$, then its projection falls strictly inside T .*

Suppose that there are at least two vertices w_1, w_2 of K_2 in $\pi^+ \cap \Theta$. If one of them, say w_1 , does not lie in B , then, by Observation 22, its projection on π falls strictly inside T , and we are done. Indeed, checking the conditions in Lemma 19 that allow $\|w_1 - w_2\| = 1$ to hold, one sees that condition 2 does not take place since the projection of w_1 falls strictly inside T , so the first condition must hold and, consequently, one of the vertices w_1, w_2 must coincide with one of the vertices of K_1 . The same reasoning apply for $\pi^- \cap \Theta$.

Now we are left with two possibilities.

(i) On both sides of the hyperplane π we have at least two points of K_2 , or all vertices of K_2 lie on one side. This case, which seems to be essential, actually has a short resolution. In this case all points from K_2 lie inside the ball B , and we are able to use some of its properties. Namely, we know that, since K_2 is a clique of size d , then the radius of the minimal ball that contains the clique equals the radius of B (even though it has a smaller dimension). This means that the center of that minimal ball must coincide with O , and all the points of K_2 must in fact lie on S (otherwise the minimal ball will have a smaller radius). This, by Observation 22, gives us that all the points of K_2 are projected inside T . We may then apply Lemma 19. The next step is to check the two conditions from the lemma that allow the equality to hold. Condition 1 gives that one of the vertices of K_2 is as well the vertex of K_1 . Condition 2 gives that w must

fall into the set $\Theta \cap \pi \cap S$. We note that that $\Theta \cap \pi$ is a Reuleaux simplex in the hyperplane π and $S \cap \pi$ is its circumscribed sphere. Thus, by Lemma 13, we conclude that $\Theta \cap \pi \cap S = \{v_1, \dots, v_d\}$. Therefore, in any case some of the vertices of K_2 must coincide with some of v_1, \dots, v_d and we are done in Case (i).

(ii) The other possibility is that exactly one vertex, say, w_1 , lies in π^+ , while the others lie in π^- . Moreover, we may assume that $w_1 \notin B$, otherwise, all the vertices of K_2 lie inside B and we argue as in the previous case.

Our treatment of this case is as follows. We start with a configuration with d -cliques K_1, K_2 without common vertices of the type described above. We perturb the first clique, obtaining a valid configuration of two simplices K'_1, K_2 without common vertices, provided that the initial configuration was valid. Thus, if we obtain a contradiction at some point, it means that the initial configuration was as well impossible.

We try to perturb the simplex K_1 so that w_1 will get to the top of the rugby ball Θ , constructed on the perturbed K_1 , or, in other words, that w_1 will form unit distances with all the (possibly perturbed) vertices v_1, \dots, v_d . Note that we do not modify K_2 . Here is the procedure. Suppose the distance between w_1 and v_1 is strictly less than 1. We rotate v_1 around the vertices v_2, \dots, v_d , which are fixed. The possible trajectory of v_1 is a circle, and we push v_1 towards π^- . Denote the image of v_1 by v' .

We stop the rotation procedure if one of the following two events happen:

Event 1. The distance between v' and w_1 is equal to 1.

Event 2. Some of w_2, \dots, w_d fall on the hyperplane π' , which is the hyperplane that passes through v', v_2, \dots, v_d .

Before analyzing these two possibilities, we have to mention two facts that we use. Denote by B' the image of B under the rotation, and similarly define Θ', S', S'_1 , and K'_1 .

The first fact is that w_1 stays inside Θ' , moreover, $w_1 \notin B'$. The first statement holds since we do not move v_2, \dots, v_d and because of Event 1. As for the second statement, the proof of it relies on the following simple observation:

Observation 23. *Consider two points x, y in \mathbb{R}^d and a hyperplane τ that passes through the middle of the segment xy and is orthogonal to it. Denote by τ^+ the closed halfspace bounded by τ and that contains x . Then for any point $z \in \tau^+$ we have $\|z - x\| \leq \|z - y\|$.*

Consider the hyperplane γ , which contains the intersection of S and S' . It passes through the vertices v_2, \dots, v_d and through the middle of the segment vv' . We can apply Observation 23 for the hyperplane γ and the centers of balls B, B' , which are obviously symmetric with respect to γ . Therefore, for the halfspace formed by γ that contains v , which we denote by γ^+ , we have $B \cap \gamma^+ \supset B' \cap \gamma^+$, while for the other (denoted by γ^-) it is the other way around. Next, one can see that $\pi^+ \cap \Theta \setminus B \subset \gamma^+$. This is due to the fact that any point from $\pi^+ \cap \Theta \setminus B$ projects from above inside the convex hull T of the vertices of the Reuleaux

simplex $\Theta \cap \pi$ (see Observation 22). We have $T \subset \gamma^+$. Since the rotation is made continuously, we may for sure assume that the angular distance between v and v' is less than 90° . Therefore, any point that is projected on π inside T from above, lies in γ^+ (see Fig. 4, where points that are projected inside T from above, lie in the shaded rectangle). From $\pi^+ \cap \Theta \setminus B \subset \gamma^+$ it follows that $\pi^+ \cap \Theta \setminus B \subset \pi^+ \cap \Theta \setminus B'$, and, consequently, $w_1 \notin B'$.

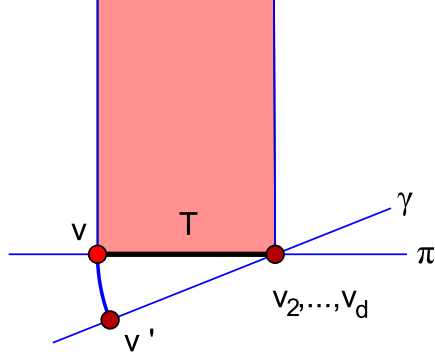


Figure 4

The second fact is that no vertex among w_2, \dots, w_d can escape from $\Theta' \cap (\pi')^-$ without falling onto π' first. This fact is proved in a similar fashion. Consider the hyperplane that contains the intersection of S_1 and S'_1 . Actually, it is again γ , moreover, due to Observation 23, we have $\Theta' \cap \gamma^- \supset \Theta \cap \gamma^-$. Indeed, since the spheres S_2, \dots, S_d do not change, we only have to look at the intersection of B_1 and B'_1 , and we fall into a situation which is similar to the one considered in the previous paragraph. Therefore, the only way for a point w_i to escape Θ' is to fall onto γ first. But this is not possible, because any position of γ was a position of π' at the earlier stage of rotation, so the point w_i has to fall onto π' first.

We go on to the analysis. Suppose that Event 2 happens, and the point w_2 from K_2 falls onto π' . Then we have two vertices of K_2 in $(\pi')^+$, and by Lemma 19 we are done unless w_2 coincides with one of the vertices of K'_1 . Moreover, we are done unless $w_2 = v'$, which means that before the rotation w_2 lay on the same arc as v' . However, the projection of this arc on the hyperplane π is a straight segment connecting the vertex v_1 and the center O . Therefore, the projection of w_2 should fall inside T , and, since there is more than one point in π^- , we are done by Lemma 19. Thus, the second event either leads us to a contradiction, if we assume that there is no common vertex of K_1 and K_2 , or gives the common vertex of K_1 and K_2 .

If Event 1 happens, then we take another vertex of K_1 and proceed in the same way. Finally, assume that we finished the procedure and have not found the common vertex of K_1 and K_2 yet. Denote by v'_1, \dots, v'_d the images of the vertices of K_1 after all the rotations. Then w_1 forms a unit d -simplex with v'_1, \dots, v'_d . In this case all the vertices $v'_1, \dots, v'_d, w_2, \dots, w_d$ lie on the unit

sphere with center in w_1 . We are left to show that a unit $(d-1)$ -simplex and a unit $(d-2)$ -simplex on a $(d-1)$ -dimensional sphere of radius greater than $1/\sqrt{2}$ must share a common vertex, which follows from the (second part of the) statement of Theorem 17 by induction.

4.3 Proof of Theorem 17. Spherical case

4.3.1 Preliminaries on spherical geometry

In what follows we will work on a d -dimensional sphere Γ of radius greater than $1/\sqrt{2}$.

Spherical geometry is very similar to Euclidean. To make the proof work in this case, one should, more or less, only change the notation: planes are changed to diametral (or great) spheres, halfspaces to hemispheres, balls to spherical caps. We will often use the Euclidean names for the spherical objects, e.g., say “a plane” instead of “a diametral sphere”. This should not cause confusion, since we will mostly work in terms of the internal spherical geometry. However, when it is convenient, we will think of the sphere as a subset of a Euclidean space, and interpret points of Γ as vectors. In the next several paragraphs we will point out some properties of spherical geometry that we will use in the proof. For an introduction to elementary spherical geometry we refer the reader to Chapter 1 of the book due to L. Fejes Tóth [12]. For a systematic treatment of spherical geometry, that by far covers all the material used in this paper we refer to [2].

1. There is a natural way to assign dimensions to spherical planes, such that the definition will work the same way as in the Euclidean case. Namely, the dimension of a diametral sphere is equal to the dimension of the minimal Euclidean plane that contains it. Note that a spherical line consists of two points.

2. For a flat (diametral sphere) γ denote by γ^* the maximal flat such that any vector from γ^* is orthogonal to any vector in γ . If γ is a hyperplane (diametral sphere of codimension 1), by γ^+, γ^- we denote the closed half-spaces bounded by γ .

3. For a given hyperplane γ and an arbitrary point $\Gamma \setminus \gamma^*$ we can define the projection of a point v to γ . Consider the two-point set γ^* . Then the projection v' of v on γ is the closest intersection point to v of the great circle that goes through γ^* and v with the plane γ .

4. We define the reflection R_γ with respect to a given hyperplane γ . For any given point v in $\Gamma \setminus \gamma^*$ we consider the great circle that contains γ^* and v , and find a point $R_\gamma(v)$ on that circle, which is symmetric to v with respect to the projection of v on γ . As for the γ^* , the reflection interchanges the two points in γ^* .

5. Using reflections, it is easy to introduce the notion of orthogonality to a hyperplane, which would be convenient for us. Namely, a plane σ is orthogonal to a hyperplane γ , if $R_\gamma(\sigma) = \sigma$.

6. Suppose that we have a k -sphere Ω on Γ , which is not diametral. It

is easy to show that any such sphere is contained in a (spherical) plane γ of dimension $(k + 2)$. Indeed, taking a Euclidean point of view, for any k -sphere there is a $(k + 1)$ -sphere with center in the center of Γ that contains Ω . This is the desired plane (of dimension one higher than the dimension of the sphere by definition). Note that this is the minimal plane that contains Ω .

7. For points in an open hemisphere Γ^+ of Γ one can easily define the distance between two points as the shorter angle between the corresponding vectors. In particular, the distance between any point in γ and any point in γ^* is $\pi/2$. We denote the spherical distance between $u_1, u_2 \in \Gamma^+$ by $\rho(u_1, u_2)$.

8. One could define an angle between the two intersecting arcs as the dihedral angle between the corresponding vector planes. For three distinct points $u_1, u_2, u_3 \in \Gamma^+$ we denote by $A(u_1, u_2, u_3)$ the angle between the arcs u_1u_2 and u_2u_3 .

9. There is a version of Pythagoras' theorem for spherical triangles. Namely, given a right spherical triangle u_1, u_2, u_3 in Γ^+ with $A(u_1, u_2, u_3) = \pi/2$, one have $\cos(\rho(u_1, u_3)) = \cos(\rho(u_1, u_2)) \cos(\rho(u_2, u_3))$. Moreover, Pythagoras' theorem is a corollary of the spherical cosine law:

$$\begin{aligned} \cos(\rho(u_1, u_3)) &= \cos(\rho(u_1, u_2)) \cos(\rho(u_2, u_3)) + \\ &\quad + \sin(\rho(u_1, u_2)) \sin(\rho(u_2, u_3)) \cos(A(u_1, u_2, u_3)). \end{aligned} \quad (1)$$

Out of this one can deduce the following statement: suppose we are given three distinct points $u_1, u_2, u_3 \in \Gamma^+$ and the angle $A(u_1, u_2, u_3)$ between the arcs u_1u_2 and u_2u_3 is at least $\pi/2$. Assume moreover, that $\rho(u_1, u_2), \rho(u_3, u_2)$ are less than $\pi/2$. Then $\rho(u_1, u_3) > \max\{\rho(u_1, u_2), \rho(u_2, u_3)\}$.

10. We need the notion of a convex hull of the set of points $\{u_1, \dots, u_k\}$. The straightforward way to define it is by using the Euclidean interpretation. It is simply the intersection of the cone formed by vectors corresponding to u_1, \dots, u_k and Γ . Note that the boundary of such a convex hull is formed by planes (diametral spheres).

4.3.2 The proof

The distance in Γ , that corresponds to Euclidean distance 1, we denote by ϕ . Note that, since in our case Γ has radius greater than $1/\sqrt{2}$, we have $\phi < \pi/2$. Suppose we are given a diameter graph on Γ , which contains two simplices K_1 and K_2 on d vertices. We consider the spherical rugby ball Θ , formed by vertices of K_1 , and the diametral sphere π that contains K_1 .

The proof stays almost the same as in the Euclidean case. We describe all the differences in what follows. All the notations are translated to this case from the Euclidean case.

First, we show that the spherical rugby ball Θ is contained in one of the open hemispheres of Γ . We denote such a hemisphere by Γ^+ . Consider the unit ball B_1 with the center in v_1 (one of the vertices of K_1). On one hand, since the radius of Γ is bigger than $1/\sqrt{2}$, B_1 is contained in the open hemisphere Γ^+ with the center in v_1 . On the other hand, surely, $B_1 \supset \Theta$.

Consider a spherical Reuleaux simplex Δ and a point x inside the (spherical) convex hull of its vertices. Then an open halfsphere with the center in x contains Δ . This is due to the fact that Δ is contained in the intersection of open halfspheres with centers in the vertices of Δ , therefore, if we think about the vector interpretation of the situation, any vector y from Δ has positive scalar products with any of the vectors representing the vertices. Since x is a convex combination of the vectors representing the vertices, x and y have positive scalar product. We formulate it as the first part of the following observation:

Observation 24. 1. Consider a spherical Reuleaux simplex Δ on Γ and a point x that lies in the spherical convex hull of the vertices of Δ . Then for any $y \in \Delta$ we have $\rho(x, y) < \pi/2$.
2. Consider any diametral hypersphere γ on Γ and any point x , $x \notin \gamma^*$. Let x' be the projection of x to γ . Then $\rho(x, x') < \pi/2$.

Lemma 19 holds for the spheres. Here is what we have to check:

I. We use the following implication extensively: consider the points $v \in \Gamma^+$ and $w \in \gamma \cap \Gamma^+$ for some plane γ . Denote by v' the projection of v to γ . If $\rho(v', w') > \rho(v', w)$ for some w' in $\gamma \cap \Gamma^+$, then $\rho(v, w') > \rho(v, w)$. This statement follows from the spherical Pythagoras' theorem (point 9 from the previous subsection) and from Observation 24.

II. The proof of the lemma in the spherical case should start from one of the following chain of inequalities: either

$$\rho(v, w) \leq \rho(v', w) < \rho(v_1, w) \leq \phi$$

or

$$\rho(v, w) \leq \rho(v, w') \leq \rho(v, v_1) \leq \phi.$$

Here v', w' are the projections of v, w to the base of the rugby ball. The second inequality in both chains is justified using the previous point (and also the next two points). In the Euclidean case the choice of one chain of inequalities depends on which point among v, w is closer to π (this gives us the first inequality in both chains). In the spherical case we can also choose one chain, since $\max\{A(v, w, w'), A(w, v, v')\} > \pi/2$, and by the last property from point 9 from the previous subsection, combined with Observation 24, $\max\{\rho(v, w'), \rho(w, v')\} \geq \rho(v, w)$. The angle inequality holds since the vertices v, v', w, w' form a quadrangle with two right angles, while the sum of the angles in a quadrangle on the sphere is more than 2π .

III. We have to verify that Lemma 21 works in the spherical case. The proof goes word for word, but we present it here with the proof for the seek of clarity.

Lemma 25. Consider a closed halfsphere ω^+ in Γ bounded by a diametral hypersphere ω . Let Υ be a sphere with center in C , where $C \in \omega$. Let Ω be an open region on Υ , $\Omega \subset \Upsilon \cap \omega^+$. Consider two points $X \in \omega^+$, $Y \in \Omega$. Then one can find a point $Y' \in \partial\Omega$ such that

- $\rho(X, Y) < \rho(X, Y')$, if $X \neq C$;

- $\rho(X, Y) = \rho(X, Y')$, if $X = C$.

Proof. The equality from the statement of the lemma is obvious since Ω and $\partial\Omega$ both lie on Υ . As for the inequality, consider the two-dimensional diametral sphere γ that contains the points C, X, Y . The diametral circle CY splits γ into two closed halvespheres γ^+, γ^- . Let $X \in \gamma^+$. There is at least one point $Y' \in \partial\Omega$ in γ^- , which is different from Y . Then we have the inequality for the angles $A(X, C, Y) < A(X, C, Y')$ and, by the spherical law of cosines (1) we get that $\rho(X, Y) < \rho(X, Y')$. This is because the first summand in (1) stays the same in both cases, while in the second one the only change is that the last multiple gets smaller. Note that we do not need to put any restrictions on $\rho(X, C), \rho(C, Y)$, as it is done in the end of point 9 of the previous subsection. \square

IV. The rest of the proof of the spherical version of Lemma 19 goes the same, since the projections of the set T behave the same as in the Euclidean case.

Having finished the description of the changes in the proof of Lemma 19 in the spherical case, we return to the proof of the theorem. As in the Euclidean case, spheres $S \cap S_i$ lie in the hyperplane π_i , which is orthogonal to π . The first thing we note is that $S \cap S_i$ is a $(d-2)$ -sphere that is not diametral. It would be diametral only if both S and S_i are diametral, which is not the case. Thus, by point 6 from the previous subsection, the minimal plane that contains $S \cap S_i$ is of codimension 1. Next, note that $R_\pi(S \cap S_i) = S \cap S_i$. This is due to the fact that both centers of S and S_i lie in π . Hence, the same should hold for π_i , and by point 5 from the previous subsection we obtain the desired orthogonality.

Combining the spherical version of Lemma 19 and the fact that π_i is orthogonal to π with the Euclidean proof, we almost come to the last case in which we have one vertex of K_2 in $\pi^+ \cap \Theta$, while the rest lie in $\pi^- \cap \Theta$.

However, we should make sure that the circumscribed ball considerations still work in the spherical case. Indeed, suppose we are given two unit simplices K_1, K_2 on d vertices. Suppose K_2 lies inside the d -dimensional ball B of diameter f , which is a ball of minimal diameter that contains K_1 . Denote by B_2 the circumscribed ball for K_2 . Then, if B and B_2 do not coincide, the intersection $B \cap B_2$ is contained in a ball of smaller radius. Indeed, if we choose an arbitrary point u on the open segment connecting the centers O, O_2 of B and B_2 , respectively, then by the last property in point 9 from the previous subsection, combined with the fact that $\rho(x, u), \rho(u, O), \rho(u, O_2) < \pi/2$ for any $x \in B \cap B_2$, we have $\rho(x, u) < \max\{\rho(x, O), \rho(x, O_2)\}$, since either $A(O, u, x)$ or $A(O_2, u, x)$ is at least $\pi/2$. We obtain that K_2 is contained in a ball of radius strictly smaller than f , which is impossible. Thus, the centers of B and B_2 coincide, and the rest of the argument in this case works exactly as for the Euclidean space.

Returning to the last part of the proof, one can translate it word for word from the Euclidean case. We note that the plane γ satisfies the following equation: $R_\gamma(B') = B$, which is why $B \cap \gamma^+ \supset B' \cap \gamma^+$. Indeed, $B' \cap \gamma^+ = R_\gamma(B \cap \gamma^-)$ and $B \cap \gamma^-$ is less than a halfball, while $B \cap \gamma^+$ is bigger than a halfball. Similar reasoning applies for the inclusion $\Theta' \cap \gamma^- \supset \Theta \cap \gamma^-$.

Finally, consider the case when w_2 falls into π' and, moreover, w_2 coincides with v' . To conclude the proof, we need to check that the arc on which v', v_1 lie projects inside the spherical convex hull of vertices v_1, \dots, v_d . This is equivalent to the statement that the circle $S_2 \cap \dots \cap S_d$ and the sphere S touch at v_1 (in the plane π). In that case the circle would lie in the exterior of the ball B . The fact about touching is clear, so the point w_2 lie in $\Theta \setminus B$, and we can apply the spherical analogue of Lemma 19. The proof is complete.

5 Acknowledgement

We are grateful to Filip Morić and János Pach for several interesting and stimulating discussions on the problem, Eyal Ackerman for pointing out the paper [13], Ido Shahaf for his valuable remarks on the presentation of the paper and for providing us with Figure 2, to Arseniy Akopyan and Roman Karasev for valuable discussions and for bringing Kirschbraun's theorem to our attention, and anonymous referees for numerous valuable comments that helped to significantly improve the presentation of the paper.

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